

Microlocal Analysis and Renormalization in Finite Temperature Field Theory

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Received May 11, 2006; accepted June 22, 2006
Published Online: January 4, 2007

We reassess the problem of renormalization in finite temperature field theory (FTFT). A new point of view elucidates the relation between the ultraviolet divergences for $T = 0$ and $T \neq 0$ theories and makes clear the reason why the ultraviolet behavior keeps unaffected when we consider the FTFT version associated to a given quantum field theory (QFT). The strength of the derivation one lies on the Hörmander's criterion for the existence of products of distributions in terms of the wavefront sets of the respective distributions. The approach allows us to regard the FTFT both imaginary and real time formalism at once in a unified way in the contour ordered formalism.

KEY WORDS: finite temperature; renormalization; wavefront Set.
Mathematics Subject Classification. 35A18, 35A20, 46S60

PACS numbers: 11.10.Gh, 11.10.Wx, 11.25.Db
To Prof. Olivier Piguet

1. INTRODUCTION

As it occurs in QFT, the FTFT also exhibit ultraviolet divergences. The problem of how to make sense out of the physical meaning behind the divergences in a mathematically proper way was satisfactorily solved by the known renormalization procedure. There are some well established prescriptions currently used in QFT to attribute meaning to the initially divergent terms of the perturbation series associated to the quantities of interest. The latter can however be defined only up to certain renormalization ambiguities which, in principle, can be determined from physical reasonings. In facing the distinctions between the FTFT and QFT

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propagators, some questions take place. Once the divergences are related to certain ill-defined products of distributions, the FTFT propagator might imply changes in the conditions for the existence of the products and introduce a temperature-dependent renormalization problem. The ambiguities of the renormalization procedure associated to the physical parameters could then exhibit qualitative changes due to the temperature-dependence. Further, it could also change the asymptotic divergent behavior and consequently the amount of arbitrariness involved. The FTFT propagator being separable into temperature-dependent and -independent pieces, causes the mixing of the divergences and temperature-dependent terms in crossing products in the higher order terms of the perturbation expansion. Depending on the renormalization procedure adopted some of those facts can become not clear. On physical grounds, one cannot expect that the differences would have fundamental consequences to the UV behavior because it arises from the short distance limit which is unaffected by the temperature once the thermal part of the propagator has support on the mass shell and decays rapidly with growing momentum because of the Bose-Einstein (or Fermi-Dirac) distribution function. This question is not really new and has been investigated by many authors using various techniques, each one putting emphasis on different aspects of the problem. Let us mention for instance the TFD proof (Matsumoto et al., 1984), the RFT method (Niemi and Semenoff, 1984), the BPHZ momentum space subtraction procedure (Gomes and Köberle, 1977; Landsman and Weert, 1987) besides the framework of axiomatic quantum field theories at finite temperature (Steinmann, 1995). More recently, it has been given by Kopper *et al.* (2001) a rigorous proof of the renormalizability of the massive ϕ_4^4 theory at finite temperature based in the framework of Wilson's flow equations, to all orders of the loop expansion.

In this article, we take the opportunity to shed a new light on the series of studies by approaching the problem from a central aspect of the renormalization which lies on the lack of definition of the distributional product in some particular context present in the perturbation series. The analysis is done under the light of a systematic use of the ideas and notions of the distribution theory. Microlocal methods of distributions in x -space are used in order to characterize the singular spectrum in terms of the wavefront set of the propagators and to determine a sufficient condition for the existence of such products (Hörmander, 1990). The asymptotic behavior of products of distributions near the singular points is evaluated by calculating the scaling degree and the singular order of distributions (Hollands and Wald, 2001, 2002; Steinmann, 1971) which govern the amount arbitrariness present in the renormalization procedure. The whole apparatus provide us with all the information we need in order to formulate the renormalization as the well posed mathematical problem of the extension of products of distributions to coincident points (Popineau and Stora, 1982; Brunetti and Fredenhagen, 2000). One shows that the divergences found in FTFT are, in fact, of the same nature as those ones in QFT. At each order, the problem of the extension in FTFT is shown to

reduce to the analogous one of the ordinary QFT. As a consequence, it is proved that the amount of arbitrariness in the renormalization procedure, as well as the type of the ambiguities remain the same when passing from a given QFT to the associated FTFT version. More important, our analysis allows to investigating the issue, as much as possible, in a model independent way and free from the technical difficulties of thermal loop calculations common to the various conventional approaches both in ITFs and RTFs. Moreover, it allows to adopting the generalized unified framework of the contour ordered formalism (COF) considering at a time both ITF and RTF.

The outline of the article is as follows. We begin in Section 2 by describing some basics on the microlocal analysis of singularities where the wavefront set of a distribution is introduced together with a sufficient condition for the existence of products of distributions based on its wavefront set. In Section 3, we reproduce a derivation of the FTFT free two-point function, and we prove the required wavefront set properties of this two-point function, in order to be able to insert it in the renormalization scheme. This section also includes a discussion of some examples. The extent of the analysis of these examples is to indicate where extra singular terms may come into the picture—over and those above appearing in usual QFT at $T = 0$. Section 4 contains the final considerations.

2. MICROLOCAL STUDY OF SINGULARITIES

The UV divergences are a QFT inherent problem, because the fields, as well as its correlation functions, having distributional character are defined on a continuous space-time. The perturbation expansions in QFT are made of the product of such distributions. However, products of distributions with overlapping singularities are in general not well-defined. Hence, it becomes convenient to shed some light on the problem of finding the conditions under which one has or not a well-defined product of distributions. Among the distributional analysis techniques, the framework of the *microlocal analysis* (Hörmander, 1990) is fairly suitable for the study of the UV divergences. The term microlocal analysis refers to a set of techniques of relatively recent origin which have turned out to be particularly useful in analyzing partial differential equations with variable coefficients, including those of particular interest to quantum field theory. In what follows, we shall describe an analytical method which provides sufficient conditions for the existence of the product of distributions based on the concept of the *wavefront set* (WFS) of a distribution f , denoted by $WF(f)$. It is a refined description of the singularity spectrum. More important, WFS not only describes the set where a distribution is singular, but also localizes the frequencies that constitute these singularities. Similar notion was developed in some versions by Sato (Sato, 1969) and Iagolnitzer (Iagolnitzer, 1975). The present definition is due to Hörmander (Hörmander, 1990) who has made use of this terminology due to an existing

analogy between the “propagation” of singularities of distributions and the classical construction of propagating waves by Huyghens.

Let f be a distribution on an open set $X \subset \mathbb{R}^d$; then the *singular support* of f is the complement of the largest relatively open subset X^1 of X whereon f is *smooth* ($f|_{X^1} \in C_0^\infty$). A point x_0 is said to be a *non-singular point* of a distribution f if there exists a cutoff function $\phi \in C_0^\infty(V)$, with support in some neighborhood V of x_0 , such that the Fourier transform

$$\widehat{f\phi}(k) = \int d^d x f(x)\phi(x)e^{ikx} ,$$

is of fast decrease for all directions $k \in \mathbb{R}^d$. By a fast decrease in the k direction of $\widehat{f}(k)$, one must understand that there is a constant C_N , for all $(N = 1, 2, 3 \dots)$, such that $(1 + |k|)^N |\widehat{f}(k)| \leq C_N$ remains bounded. In particular, if x_0 is a singular point of the distribution f , and $\phi \in C_0^\infty(V)$ is such that $\phi(x_0) \neq 0$; then ϕf is also of compact support and singular in x_0 . In this case, can still occur some directions in k -space over which $\widehat{\phi f}$ is asymptotically bounded. A direction k for which the Fourier transform $\widehat{f}(k)$ of $f(x) \in \mathcal{D}(V)$ shows to be of fast decrease is called to be a *regular direction* of $\widehat{f}(k)$. This suggests that we can single out singular directions as well as singular point, and for the establishment of these concepts only the behavior of f and of \widehat{f} restricted to an arbitrarily small neighborhood of the singular point x_0 is relevant.

Let $f(x)$ be an arbitrary distribution not necessarily of compact support on an open set $X \subset \mathbb{R}^d$. Then, the set of all pairs composed first by the its singular points $x \in X$ and second by the associated nonzero singular directions k ,

$$WF(f) = \{(x_0, k) \in X \times (\mathbb{R}^d \setminus 0) \mid k \in \Sigma_x(f)\} , \tag{2.1}$$

is called **wavefront set** of f . The $\Sigma_x(f)$ is defined to be the complement of the set of all $k \in \mathbb{R}^d \setminus 0$ with respect to $\mathbb{R}^d \setminus 0$, for which there is an open conic neighborhood M of k such that $\widehat{\phi f}$ is of fast decrease on M . In short, to determine whether (x_0, k) is in WFS of f one must first to localize f around x_0 , to next obtain Fourier transform \widehat{f} and finally to look at the decay in the direction k .

Example 1. A small “point” scatterer on \mathbb{R} .

$$V(x) = \delta(x) \propto \int d^d x \mathbf{1} e^{-ikx} ,$$

i.e., $\widehat{V} = \mathbf{1}$ does not decay in any direction k : $WF(\delta) = \{(0, k) \mid k \neq 0\}$ has singularities in all directions.

Remark 1. We now collect some basic properties of the WFSs:

1. The $WF(f)$ is conic in the sense that it remains invariant under the action of dilatations, *i.e.* when one multiplies the second variable by a positive scalar. If $(x, k) \in WF(f)$ then $(x, \lambda k) \in WF(f)$ for all $\lambda > 0$.
2. From the definition of the wavefront set, it follows that the projection onto the first coordinate $\pi_1(WF(f)) \rightarrow x$, consists of those points that have no neighborhood where u is a smooth function, and the projection onto the second coordinate $\pi_2(WF(f)) \rightarrow \Sigma_x(f)$, is the cone around k attached to a such point denoting the set of high-frequency directions responsible for the appearance of a singularity at this point.
3. The WFS of a smooth function is the empty set.
4. For all smooth function ϕ with compact support $WF(\phi f) \subset WF(f)$.
5. For any partial linear differential operator P , with C^∞ coefficients, one has

$$WF(Pf) \subseteq WF(f) .$$

6. If f and g are two distributions belonging to $\mathcal{D}'(\mathbb{R}^d)$, with wavefront set $WF(f)$ and $WF(g)$, respectively; then the wavefront set of $(f + g) \in \mathcal{D}'(\mathbb{R}^d)$ is contained in $WF(f) \cup WF(g)$.

In the perturbation scheme of quantum field theories, one finds formal operations on distributions which can be in general not well-defined. In order to give precise statements on the existence of the product of these distributions, we appeal to a criterion based on the WFS of the distributional factors the so-called **Hörmander’s Criterion**. Let u and v be distributions; if the WFS of u and v are such that the following direct sum

$$WF(u) \oplus WF(v) \stackrel{\text{def}}{=} \{(x, k_1 + k_2) \mid (x, k_1) \in WF(u), (x, k_2) \in WF(v)\} , \quad (2.2)$$

does not contain any element of the form $(x, 0)$, then the product uv there exists and $WF(uv) \subset WF(u) \cup WF(v) \cup (WF(u) \oplus WF(v))$. Hence, the product of the distributions u and v is well-defined around x , if u , or v , or both distributions are regular in x . Otherwise, if u and v are singular in x , the product can still exist if the sum of the second components from $WF(u)$ and $WF(v)$ related to x can be linearly combined with nonnegative coefficients to vanish only by a trivial manner.

Example 2. The distributions $u, v \in \mathcal{D}'(\mathbb{R})$, $u(x) = \frac{1}{x+i\epsilon}$ and $v(x) = \frac{1}{x-i\epsilon}$, with the Heavyside distributions $\widehat{u}(k) = -2\pi i\theta(-k)$ and $\widehat{v}(k) = 2\pi i\theta(k)$ as their Fourier transforms, have the following WFSs:

$$WF(u) = \{(0, k) \mid k \in \mathbb{R}^- \setminus 0\} , \quad WF(v) = \{(0, k) \mid k \in \mathbb{R}^+ \setminus 0\} .$$

Thus, from the Hörmander’s Criterion one finds that there exist the powers of u^n and v^n . However, the product between u and v do not match the criterion above and do not exist. This example clearly indicates that one can multiply distributions

even if they have overlapping singularities, provided their WFSs are in favorable positions. Such an observation is significant because it makes clear that *the problem is not only where the support is, but in which directions the Fourier transform is not rapidly decreasing!*

Example 3. The Feynman propagator for massive scalar field

$$\Delta_F(x) \stackrel{\text{def}}{=} \theta(x^0)\Delta_+(x; m^2) - \theta(-x^0)\Delta_-(x; m^2), \tag{2.3}$$

can have its WFS constitution studied from the WFS of the Wightman functions,

$$\begin{aligned} WF(\Delta_{\pm}) &= \{((0, \mathbf{0}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k}) \mid (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, \lambda \in \mathbb{R}_+\} \\ &\cup \{((|\mathbf{x}|, \mathbf{x}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k}) \mid \mathbf{x}, (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, \lambda \in \mathbb{R}_+\}, \tag{2.4} \\ &\cup \{((-|\mathbf{x}|, \mathbf{x}); (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k}) \mid \mathbf{x}, (\mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, \lambda \in \mathbb{R}_+\}, \end{aligned}$$

and from the WFS of $\theta(\pm t \mp t') \stackrel{\text{def}}{=} \theta^{\pm}$,

$$WF(\theta^{\pm}) = \{((0, \mathbf{x}); (\pm\lambda k_0, \mathbf{0}) \mid \mathbf{x} \in \mathbb{R}^3, \mathbf{k}_0 \in \mathbb{R}, \lambda \in \mathbb{R}_+\}. \tag{2.5}$$

One can easily conclude that is not possible to form a non trivial linear combination with non-negative coefficients in order to produce a vanishing second component in the direct sum of the WFSs above. So,

$$(x, 0) \notin WF(\theta^{\pm}) \oplus WF(\Delta_{\pm}). \tag{2.6}$$

Therefore, from the Hörmander’s criterion, the Feynman propagator can be well-defined in terms of the product above and

$$WF(\theta^{\pm} \cdot \Delta_{\pm}) \subset WF(\theta^{\pm}) \cup WF(\Delta_{\pm}) \cup (WF(\theta^{\pm}) \oplus WF(\Delta_{\pm})). \tag{2.7}$$

However, in the powers $(\Delta_F)^n$ there exist products like $\Delta_+\Delta_-$ and from (2.4), one can see that $(x, 0) \in WF(\Delta_+) \oplus WF(\Delta_-)$ and it occurs for the singular point $x = 0$. In this sense, one must be careful when manipulating such products. In fact, they are known to exist anywhere, except at $x = 0$. Such an ill-definition, manifested as divergences, requires the treatment of the renormalization. Notice further that

$$(x, 0) \notin WF(\Delta_{\pm}) \oplus WF(\Delta_{\pm}). \tag{2.8}$$

In particular, it can be used

$$\Delta_{\pm}(x; m^2) = \frac{\pm i}{(2\pi)^3} \int d^4k_1 \theta(\pm k_1^0) \delta(k_1^2 - m^2) e^{-ik_1x},$$

and $\widehat{\Delta}_{\pm}(k_1, k_2) = \pm i(2\pi)^4 \delta(k_1 + k_2) \theta(\pm k_1^0) \delta(k_1^2 - m^2)$ as a representation of the Fourier transform, to verify that the wavefront set of Feynman propagator has the

following covariant form (Radzikowski, 1996a,b):

$$WF(\Delta_F) = \{(x_1, k_1); (x_2, k_2) \in (\mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \setminus 0) \mid x_1 \neq x_2, (x_1 - x_2)^2 = 0, \\ k_1 \parallel (x_1 - x_2), k_1 + k_2 = 0, k_1^2 = 0, k_1^0 > 0 \text{ if } x_1 \succ x_2 \text{ and} \\ k_1^0 < 0 \text{ if } x_1 \succ x_2\} \cup \{(x_1, k_1); (x_2, k_2) \in (\mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \setminus 0) \mid x_1 = x_2, \\ k_1 + k_2 = 0, k_1^2 = 0\},$$

where we have used the notation that $x_1 \succ x_2$ if $x_1 - x_2$ is in the convex hull of the forward lightcone and $x_2 \succ x_1$ if $x_1 - x_2$ is in the convex hull of the backward lightcone. Notice that the condition $k_1^0 > 0$ if $x_1 \succ x_2$ and $k_1^0 < 0$ if $x_1 \succ x_2$ in $WF(\Delta_F)$ ensures the existence of products of Feynman propagators at all points away from diagonal, while these products do not satisfy the Hörmander’s criterion for multiplication of distributions over the points of the diagonal, since the sum of the second components of the WFS on the diagonal can add up to zero.

3. RENORMALIZATION OF DISTRIBUTIONS IN FTFT

In order to study the structure of the renormalization scheme in FTFT, we turn to the analysis of distributions and their products present in the perturbation series. The existence of such products are checked out via the Hörmander’s criterion based on its WFSs. Keeping in mind the renormalization procedure as an extension problem (Brunetti and Fredenhagen, 2000; Popineau and Stora, 1982) together to the its inherent arbitrariness governed by scaling degree and singular order of distributions (Brunetti and Fredenhagen, 2000; Steinmann, 1971), the perturbation expansion is further discussed. Without loss in generality, let us consider the case of a single, scalar field $\phi(x)$ in FTFT associated to spinless particles with mass $m > 0$,⁴ whose propagator is given by:

$$G^c(x, x') = \theta_c(t - t')\langle \widehat{\phi}(x)\widehat{\phi}(x') \rangle + \theta_c(t' - t)\langle \widehat{\phi}(x')\widehat{\phi}(x) \rangle . \quad (3.1)$$

The brackets $\langle \cdot \cdot \cdot \rangle$ stand for statistical average related to states of a complete orthogonal basis in Fock space. The index “c” accounts for the contour ordering in the complex time plane-t ($t = x_0 + ix_4$) whose the imaginary and real parts are interpreted to be the inverse temperature and actual time respectively. For the contour ordering prescription given by $\theta_c(t - t')$, it is supposed that the contour “c” is monotonically increasing and regular, parameterized by a parameter $\tau \in \mathbb{R}$ $C = \{t \in \mathbb{C} \mid \text{Re } t = x_0(\tau), \text{Im } t = x_4(\tau), \tau \in \mathbb{R}\}$ and $\theta_c(t - t') = \theta(\tau - \tau')$. The spectral decomposition of $\widehat{\phi}$ in terms of plane waves has the ordinary form

⁴The generalization of the present prescription to any field with arbitrary spin is straightforward.

(Landsman and Weert, 1987):

$$\widehat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} [a_k e^{-ikx} + a_k^\dagger e^{ikx}], \quad (3.2)$$

where $k_0 = \omega_k = (|\mathbf{k}|^2 + m^2)^{1/2}$. For compatibility with FTFT, we must include the statistical distribution of the particles associated. For a single, scalar field $\phi(x)$ we introduce the Bose-Einstein statistic given by $N(k_0) = [e^{\beta k_0} - 1]^{-1}$. In this case, combinations of creation and annihilation operators are given by (Landsman and Weert, 1987):

$$\begin{aligned} \langle a_k^\dagger a_k \rangle &= (2\pi)^3 2\omega_k N(\omega_k) \delta(\mathbf{k} - \mathbf{k}') \\ \langle a_k a_k^\dagger \rangle &= (2\pi)^3 2\omega_k [N(\omega_k) + 1] \delta(\mathbf{k} - \mathbf{k}'), \end{aligned} \quad (3.3)$$

with the combinations of two creation or two annihilation operators vanishing. The correlation functions $C^>(x, x') = \langle \widehat{\phi}(x) \widehat{\phi}(x') \rangle = C^<(x', x)$ turns to have the following spectral expansion

$$\langle \widehat{\phi}(x) \widehat{\phi}(x') \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \rho(k) [1 + N(k_0)],$$

where $\rho(k) = 2\pi [\theta(k_0) - \theta(-k_0)] \delta(k^2 - m^2)$. Their Fourier transforms, related by $\widetilde{C}^<(k) = \rho(k) [N(k_0) + 1] = e^{\beta k_0} \widetilde{C}^>(k)$, can be used in order to write the contour ordered propagator in the form (Landsman and Weert, 1987):

$$G^c(x, x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \rho(k) [\theta_c(t - t') + N(k_0)]. \quad (3.4)$$

Another useful form is obtained after integration on k_0 ,

$$\begin{aligned} G^c(x, x') &= \theta_c(t - t') \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} \left\{ [N(\omega_k) + 1] e^{-ik(x-x')} + N(\omega_k) e^{ik(x-x')} \right\} \\ &+ \theta_c(t' - t) \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} \left\{ [N(\omega_k) + 1] e^{ik(x-x')} + N(\omega_k) e^{-ik(x-x')} \right\}. \end{aligned} \quad (3.5)$$

Although each possible contour would correspond to a specific formalism of FTFT, there are restrictions on the contours due to the necessary analyticity of the correlation functions and the KMS condition (Landsman and Weert, 1987). These conditions cause the support of the two point function to be analytic on the strip given by $-\beta \leq \text{Im}(t - t') \leq \beta$, which on the closure the distributional character takes place. Furthermore, for the analyticity of $C^>(x, x')$, which because of the factor $\theta_c(t - t')$ has vanishing contributions to the propagator (3.1) if t' succeeds t on C , it is required that $-\beta \leq \text{Im}(t - t') \leq 0$. Conversely, for the analyticity of $C^<(x, x')$, with factor $\theta_c(t' - t)$, it is required that $0 \leq \text{Im}(t - t') \leq \beta$. Combining

both relations one can conclude that if the complex time t_1 succeeds t_2 on C , then there follows that $Im t_2 \geq Im t_1$ which imposes that C must have a non increasing imaginary part. In other words, C must have constant or decreasing imaginary part. This is called the *monotonicity condition*.

At this point, once the adopted approach does not depend on Feynman graphics calculations, we can proceed the analysis without the need in specializing to Minkowskian RTF or Euclidean ITF parameterizations of the contour. From (3.5) we select two typical distributions a temperature dependent piece, $G_{\text{mat}}^{(c)\pm}$, and a temperature independent piece, $G_{\text{vac}}^{(c)\pm}$, whose labels refer to *matter piece* and *vacuum piece*, respectively:

$$G_{\text{mat}}^{(c)\pm}(x, x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} N(\omega_k) e^{\mp ik(x-x')},$$

$$G_{\text{vac}}^{(c)\pm}(x, x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} e^{\mp ik(x-x')}.$$

In terms of these distributions, the general contour propagator turns to be

$$G^c(x, x') = \theta_c(t - t') \left[G_{\text{mat}}^{(c)+}(x, x') + G_{\text{vac}}^{(c)+}(x, x') + G_{\text{mat}}^{(c)-}(x, x') \right]$$

$$+ \theta_c(t' - t) \left[G_{\text{mat}}^{(c)-}(x, x') + G_{\text{vac}}^{(c)-}(x, x') + G_{\text{mat}}^{(c)+}(x, x') \right].$$

(3.6)

The structure of the propagators of FTFT suggests that, at a given order in perturbation series, the crossing products between matter and vacuum pieces would produce qualitatively different divergences. Furthermore, one could expect it to have also a proliferation of divergent terms. Another possible distinction between FTFT and QFT version would be on the amount of arbitrariness through the contribution to the singular order besides the establishment of a temperature dependent renormalization extension problem. We are going to verify that in some sense the fact above does occur. The perception of either of these points and their consequences would become more difficult or not depending on the renormalization procedure adopted.

We now turn to investigate the divergent content of the distributions $G_{\text{mat(vac)}}^{(c)\pm}$ by calculating their WFSs.

Theorem 1. *Only the temperature-independent part contributes to the WF (G^c).*

Proof: We proceed the prove using the stationary phase method (see for example (Reed and Simon, 1975), Section IX.10). The phases of the distributions above have all the same form $\mp ik(x - y)$. It is useful to unify the notation as much as

possible and represent them all by defining the following integral

$$G_{\text{mat(vac)}}^{(c)\pm} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\tilde{f}_{\text{mat(vac)}}(\mathbf{k}; m^2, \beta)}{2\omega_k} e^{\mp i[\omega(t-t') - \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')]} , \quad (3.7)$$

where $\tilde{f}_{\text{mat}}(\mathbf{k}; m^2, \beta) = N(\omega_k)$ and $\tilde{f}_{\text{vac}}(\mathbf{k}; m^2, \beta) = 1$. One can define the phase function φ_{\pm} ,

$$\varphi_{\pm}(\mathbf{k}, x - x') = \pm [(t - t')|\mathbf{k}| - (\mathbf{x} - \mathbf{x}') \cdot \mathbf{k}] , \quad (3.8)$$

to obtain the following oscillatory integrals for the distributions:

$$G_{\text{mat(vac)}}^{(c)\pm} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} a_{\pm\text{mat(vac)}}(t - t', |\mathbf{k}|; m^2) e^{-i\varphi_{\pm}(\mathbf{k}, x - x')} . \quad (3.9)$$

where

$$a_{\pm\text{mat(vac)}}(t - t', |\mathbf{k}|; m^2) = \frac{\tilde{f}_{\text{mat(vac)}}(\mathbf{k}; m^2, \beta)}{2\omega_k} e^{\mp i[(\omega - |\mathbf{k}|)(t - t')]} \quad (3.10)$$

is the asymptotic symbol. From the definition of the phase function (3.8), one can easily see that it must be such that $\text{Im}(t - t') \leq 0$. Then, had the monotonicity condition not previously selected the possible contours, the φ_{\pm} would be ill-defined. Both are in fact manifestations of the necessary analyticity of the Green functions. The directions along which the phase in the integrand do not vary satisfying $\partial_{\mathbf{k}}\varphi_{\pm} = 0$ give us the following critical set,

$$\begin{aligned} \mathcal{C}_{\varphi_{\pm}} = & \{(x - x' = (0, \mathbf{0}), k) \mid (k \neq 0) \in \mathbb{R}^4\} \\ & \cup \{(x - x', k) \mid (\mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, (t - t') \in \mathbb{C}, \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') > 0, \\ & \quad \text{Re}(t - t') = |\mathbf{x} - \mathbf{x}'|, \text{Im}(t - t') = 0\} \\ & \cup \{(x - x', k) \mid (\mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, (t - t') \in \mathbb{C}, \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') < 0, \\ & \quad \text{Re}(t - t') = -|\mathbf{x} - \mathbf{x}'|, \text{Im}(t - t') = 0\} . \end{aligned}$$

Though there is the restriction to those terms in (3.6) which satisfy the monotonicity condition, from the additional condition $\text{Im}(t - t') = 0$, one can see that there are no contributions coming from the pieces of the contour with non-vanishing imaginary part. It has important consequences in the analysis of the WFS for the ITFs. Because the set of singular points of the WFS is a subset of $\mathcal{C}_{\varphi_{\pm}}$, and the ITF-like pieces of the contour are such that $\text{Im}(t - t') > 0$, one can conclude that the WFS associated to ITF correlation functions are empty. The stationary phase manifold Λ_{φ} is the set of points of the critical set having the non vanishing four momentum component given by the gradients $\partial_{\mu}\varphi_{\pm} = (|\mathbf{k}|, -\mathbf{k})$

and $\partial_\mu \varphi_- = (-|\mathbf{k}|, \mathbf{k})$. Then,

$$\begin{aligned} \Lambda_{\varphi_\pm} = & \left\{ (x - x' = (0, \mathbf{0}), (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid \mathbf{k} \neq \mathbf{0} \in \mathbb{R}^3, \lambda \in \mathbb{R}_+ \right\} \\ & \cup \left\{ (x - x', (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid (\mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, (t - t') \in \mathbb{C}, \right. \\ & \left. \lambda \in \mathbb{R}_+, \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') > 0, \operatorname{Re}(t - t') = |\mathbf{x} - \mathbf{x}'|, \operatorname{Im}(t - t') = 0 \right\} \\ & \cup \left\{ (x - x', (\pm\lambda|\mathbf{k}|, \mp\lambda\mathbf{k})) \mid (\mathbf{x} - \mathbf{x}' \parallel \mathbf{k} \neq \mathbf{0}) \in \mathbb{R}^3, (t - t') \in \mathbb{C}, \right. \\ & \left. \lambda \in \mathbb{R}_+, \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') < 0, \operatorname{Re}(t - t') = -|\mathbf{x} - \mathbf{x}'|, \operatorname{Im}(t - t') = 0 \right\}. \end{aligned} \tag{3.11}$$

The result above can be interpreted as the set of pairs of which the critical character of the phase is such that it breaks certain natural tendency of the integrals to converge due to its oscillatory character (see Riemann-Lebesgue Lemma (Reed and Simon, 1975)). Such pairs are, therefore, suspect to be responsible for some bad behavior of the oscillatory integral. This implies that $WF(G^{(c)\pm}) \subseteq \Lambda_{\varphi_\pm}$ (again, see Section IX.10 in (Reed and Simon, 1975)). Because we are still able to save the convergence in some or even in all those critical directions, there remains to be studied the contributions of the asymptotic symbols, $a_{\pm\text{mat}(\text{vac})}$ and, in particular of $\tilde{f}_{\text{mat}(\text{vac})}$, to the convergence of the integrals. For the temperature dependent part, to every possible contribution considered in the stationary phase manifold (3.11), the exponential factor $e^{\beta\omega_k}$ in the denominator of the integrand $\tilde{f}_{\text{mat}}(\mathbf{k}; m^2, \beta) = N(\omega_k)$ assures the condition for a fast decreasing function (see Sec. 2) to be fulfilled in every of those critical directions. This guarantees the existence of the oscillatory integral and characterizes $G_{\text{mat}}^{(c)\pm}$ to be a smooth function. Its WFS contribution is then empty. However, in the case of the vacuum piece, $\tilde{f}_{\text{vac}}(\mathbf{k}; m^2, \beta) = 1$, the factor $\frac{1}{\omega_k}$ does not suffice to assure the asymptotic fast decrease in none of those critical directions. So, every pair in Λ_{φ_\pm} turns to be an element of the WFS. Therefore we have

$$WF(G_{\text{vac}}^{(c)\pm}) = \Lambda_{\varphi_\pm} \quad \text{and} \quad WF(G_{\text{mat}}^{(c)\pm}) = \emptyset. \tag{3.12}$$

Hence, there are no contributions coming from the matter temperature-dependent part to the $WF(G^c)$. □

It perhaps is necessary to emphasize that the $G_{\text{vac}}^{(c)\pm}$ has exactly the same singular spectrum as the Wightman function Δ_\pm , in Eq. (2.4), for of the ordinary QFT. Thus, we have settled that

$$WF(G_{\text{vac}}^{(c)\pm}) = WF(\Delta_\pm). \tag{3.13}$$

There follows then the same rules discussed for Δ_{\pm} , in particular, for the product $\theta^{\pm} \cdot G_{\text{vac}}^{(c)\pm}$ one has

$$WF(\theta^{\pm} \cdot G_{\text{vac}}^{(c)\pm}) = WF(\theta^{\pm} \cdot \Delta_{\pm}) \quad \text{and} \quad (x, 0) \notin (WF(\theta^{\pm}) \oplus WF(G_{\text{vac}}^{(c)\pm})), \tag{3.14}$$

what characterizes it as well-defined and consequently, from the results of the condition of the Hörmander’s criterion (2.2),

$$\begin{aligned} WF(\theta^{\pm} \cdot G_{\text{vac}}^{(c)\pm}) &\subset WF(\theta^{\pm}) \cup WF(G_{\text{vac}}^{(c)\pm}) \cup (WF(\theta^{\pm}) \oplus WF(G_{\text{vac}}^{(c)\pm})) \\ &= WF(\theta^{\pm}) \cup WF(\Delta_{\pm}) \cup (WF(\theta^{\pm}) \oplus WF(\Delta_{\pm})). \end{aligned} \tag{3.15}$$

Because $G_{\text{mat}}^{(c)\pm}$ is a smooth function from the Property 4 in Remarks 1, the product $\theta^{\pm} \cdot G_{\text{mat}}^{(c)\pm}$ is such that

$$WF(\theta^{\pm} \cdot G_{\text{mat}}^{(c)\pm}) \subset WF(\theta^{\pm}) \quad \text{and} \quad (x, 0) \notin WF(\theta^{\pm}). \tag{3.16}$$

For this reason, in view of (3.14), (3.15) and (3.16), the FTFT contour propagator $G^{(c)}$, (3.6), is well-defined as sum of well-defined products. From the Property 6 in Remarks 1. and (2.7), we have that

$$\begin{aligned} WF(G^{(c)}) &\subset [WF(\theta^+) \cup WF(\Delta_+) \cup (WF(\theta^+) \oplus WF(\Delta_+)) \cup \\ &\quad \cup WF(\theta^-) \cup WF(\Delta_-) \cup (WF(\theta^-) \oplus WF(\Delta_-))] \supset WF(\Delta_F). \end{aligned} \tag{3.17}$$

On the other hand, in the higher orders of the perturbation calculations there arise products of propagators. In special, let us consider those terms in which there are products like $G_{\text{vac}}^{(c)+} \cdot G_{\text{vac}}^{(c)-}$. From (3.12) one can see that in the same way of the ordinary QFT for Δ_+ ,

$$(x, 0) \in WF(G_{\text{vac}}^{(c)+}) \oplus WF(G_{\text{vac}}^{(c)-}). \tag{3.18}$$

It does not match the condition for the Hörmander’s criterion. Indeed, this is also an ill-defined product if the support of the distributions include $x = 0$, what turns it to be a problem to be treated through the renormalization procedure. But products like $G_{\text{mat}}^{(c)s} \cdot G_{\text{mat}}^{(c)s'}$ and $G_{\text{mat}}^{(c)s} \cdot G_{\text{vac}}^{(c)s'}$, where $s, s' = (+, -)$, are well-defined because $G_{\text{mat}}^{(c)s}$ are smooth functions. Therefore, by considering products of propagators in the FTFT, both in RTF and ITF, one can expect that the presence of the matter piece does not contribute to generate ill-defined terms beside those already found in the ordinary QFT. Nevertheless, in the higher orders in the perturbation expansion, it appears as temperature-dependent factors to the ordinary divergences. Roughly speaking, although the ill-defined products are the same as the QFT ones, they appear with temperature-dependent factors.

Another aspect of the renormalization concerns the arbitrariness or ambiguity of the process and its relation to physical symmetries. The amount of

arbitrariness is governed by the singular order and scaling degree of the distributions involved (Steinmann, 1971)-(Brunetti and Fredenhagen, 2000). Once for $G_{\text{mat}}^{(c)\pm}$,

$$\begin{aligned} (G_{\text{mat}}^{(c)\pm})_\lambda &= G_{\text{mat}}^{(c)\pm}(\lambda(x - x'); m^2, \beta) = \int \frac{d^{(d-1)}\mathbf{k}'}{(2\pi)^3 2\omega_{k'}} N(\omega_{k'}) e^{\mp i k' \lambda(x-x')} \\ &= \lambda^{2-d} G_{\text{mat}}^{(c)\pm}(x - x'; \lambda^2 m^2, \lambda^{-1} \beta), \end{aligned}$$

then, one has $\omega \geq d - 2$, the scaling degree is $\sigma(G_{\text{mat}}^{(c)\pm}) = d - 2$ and singular order is $\Sigma(G_{\text{mat}}^{(c)\pm}) = -2$. Notice further that

$$\Sigma(G_{\text{mat}}^{c\pm}) = \Sigma(\Delta_\pm) = \Sigma(G_{\text{vac}}^{(c)\pm}). \tag{3.19}$$

Hence, for the FTFT propagator $G^{(c)}$ in (3.6), we obtain that

$$\sigma(\theta^\pm G_{\text{mat(vac)}}^{c\pm}) = \sigma(G^{(c)}) = \sigma(\Delta_F) = d - 2, \tag{3.20}$$

$$\Sigma(\theta^\pm G_{\text{mat(vac)}}^{c\pm}) = \Sigma(G^{(c)}) = \Sigma(\Delta_F) = -2. \tag{3.21}$$

We shall analyze, as a representative case of the higher order product in the perturbation series, the square of the propagator associated to the branch of the contour which is parameterized forward in the real time only. We consider again the products of propagators arising in the perturbation series. For the products like $G_{\text{mat}}^{(c)s} \cdot G_{\text{mat}}^{(c)s'}$, $G_{\text{mat}}^{(c)s} \cdot G_{\text{vac}}^{(c)s'}$ and $G_{\text{vac}}^{(c)s} \cdot G_{\text{vac}}^{(c)s'}$ we have

$$\begin{aligned} \sigma(G_{\text{mat(vac)}}^{(c)s} \cdot G_{\text{mat(vac)}}^{(c)s'}) &= 2(d - 2) \\ \Sigma(G_{\text{mat(vac)}}^{(c)s} \cdot G_{\text{mat(vac)}}^{(c)s'}) &= 2(d - 2) - d. \end{aligned} \tag{3.22}$$

The scaling degree and the singular order are the same for both the matter or vacuum pieces products. In view of this, one can see that the singular order determines the number of arbitrary coefficients (counter terms) in the renormalization procedure. As a simple example, let us examine an 1-loop diagram in $\frac{g}{4!}\phi^4$, a truncated 4-point diagram with two internal lines connecting two different vertex,

$$\begin{aligned} \Gamma^{(4)} \sim g^2 [G^{(c)}(x - x')]^2 &= g^2 \left\{ \sum_{s=+,-} \theta^s \theta^s G_{\text{vac}}^{(c)s} G_{\text{vac}}^{(c)s} + \sum_{s=+,-} \theta^s \theta^{-s} G_{\text{vac}}^{(c)s} G_{\text{vac}}^{(c)-s} \right. \\ &+ 2 \sum_{s \ s'} \theta^s \theta^s G_{\text{vac}}^{(c)s} G_{\text{mat}}^{(c)s'} + 2 \sum_{s \ s'} \theta^s \theta^{-s} G_{\text{vac}}^{(c)s} G_{\text{mat}}^{(c)s'} \\ &\left. + \sum_{s \ s' \ s''} \theta^s \theta^s G_{\text{mat}}^{(c)s'} G_{\text{mat}}^{(c)s''} + \sum_{s \ s' \ s''} \theta^s \theta^{-s} G_{\text{mat}}^{(c)s'} G_{\text{mat}}^{(c)s''} \right\}. \end{aligned} \tag{3.23}$$

Notice that the sum of products of distributions falls into different categories. As it was shown from (3.12) to (3.18) and in the chain of reasoning just after, the only term which exhibits an ill-defined product is the second one. That product is well-defined elsewhere, except at $x - x' = 0$. This is the target of the renormalization in the present case. The degree of arbitrariness is governed by (3.22) and the number of counter terms is limited by certain physical symmetries.

Next, we consider an overlapping higher loop with three internal lines connecting two vertices,⁵

$$\Gamma^{(2)} \sim g^2 [G^{(c)}(x - x')]^3 .$$

One finds an abundance of ill-defined terms as compared to the ordinary QFT case. There will appear ill-defined products like

$$\theta^s \theta^s \theta^{-s} G_{\text{vac}}^{(c)s} G_{\text{vac}}^{(c)s} G_{\text{vac}}^{(c)-s} \quad \text{and} \quad \theta^s \theta^s \theta^{-s} G_{\text{mat}}^{(c)s} G_{\text{vac}}^{(c)s} G_{\text{vac}}^{(c)-s} .$$

The former suffers from the same illness as the second term of (3.23), though it has a different degree and it is also to be treated in a temperature-independent fashion. The latter, due to the presence of a matter piece factor, in view of (3.22), could indicate a temperature-dependent renormalization problem. However, from (3.12) and (3.18), one can easily see verify that such an ill-definition is due to the product of vacuum pieces only. Notice that this term was treated in a temperature-independent way in the lower order term (3.23) and that the temperature-dependent part appears as simple factor. For the general case of the doubling of degrees of freedom with contour parameterized both forward and backward in the real time, the analysis is similar.

This quantitative analysis has shown that, despite the existence of temperature-dependent factors multiplying the ill-defined products, from the point of view of the renormalization problem, it can be treated order by order as a vacuum renormalization problem. Furthermore, the degree of arbitrariness in the process for a given order is limited by the singular order of the temperature-independent piece and, from products of them, there arise at each order an ill-defined product that is leading in singular order and degree of arbitrariness. This makes clear that the matter piece, being absent from a singular spectrum, cannot include any new contribution to FTFTs concerning the category of ill-defined products yet found in the ordinary QFT.

4. CONCLUSIONS

The results concerning the renormalization of FTFT has been extensively analyzed in the literature. The present contribution lies on the method which allows us

⁵ Others Feynman diagrams can be composed by convolutions of propagators. In essence, the presence of convolutions contribute to the well behavior of the product of distributions, by decreasing the singular order and improving the fast decay of the symbols.

to clarify some points in the comparison between QFT and FTFT renormalization. The problem of the divergences was faced from the ground by the mathematical study of the basic ill-defined products distributions, *i.e.*, the lack of definition of the distributional product on the coinciding points. An important role was played by the microlocal analysis. By using the Hörmander's criterion, based on the WFSs of distributions, we have shown that the contribution to form ill-defined products comes from the temperature-independent pieces only. Hence, the matter piece does not contribute to form divergent terms though it can appear as factors of the divergent ones. The structure of the propagators of FTFT, being separable into vacuum and matter pieces turns easy the analysis of the ill-defined products. As a matter of fact, one shows that the separation also generates an increasing on the number of ill-defined products in the perturbation series due to the mixing of these factors in crossing products in the higher order terms of the perturbation expansion. The matter piece appears then as temperature-dependent factors of ill-defined products vacuum pieces in the higher orders of the perturbation series. Focusing on the perturbation series, the degree of arbitrariness in the process for a given order is determined by the temperature-independent ill-defined product leading in singular order. Hence, the problem of the extension reduces at each order to the analogous one of the ordinary QFT. Consequently, it is proved that the amount of arbitrariness in the renormalization procedure, as well as the type of the ambiguities, if conveniently treated, remains the same when passing from a given QFT to the associated FTFT version. The perception of either of these points and their consequences could be difficult or not depending on the renormalization procedure adopted.

Applications of the results given in this paper will appear in a coming paper (Franco and Acebal), where we study the renormalization of the electromagnetic and gravitational couplings of an electron which is immersed in a heat bath under the light of the scheme of Brunetti-Fredenhagen-Holland-Wald (Brunetti and Fredenhagen, 2000; Hollands and Wald, 2001,2002), who have demonstrated renormalizability of QFTs satisfying the requirements of Weinberg's theorem on general curved space-times using a microlocal adaptation of the Epstein-Glaser approach. Our aim is to clarify the connection between microlocal analysis and the area of QFT at a finite temperature.

ACKNOWLEDGMENTS

J. L. Acebal was supported by the Brazilian agency Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

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